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Note

An extremal problem of orthants containing at most one point besides the origin☆

Yoshiyasu Ishigami[1]

Department of Mathematics, Waseda University, 3-4-1 Okubo, Shinjuku, Tokyo 169, Japan

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Abstract

Let d, n be positive integers, and P a set of n points in the d-dimensional Euclidean space. For $x \in P$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \in \{-1, 1\}^d$, the εth-orthant of x is defined by $L_d(x, \varepsilon) := \{x + y \mid y = (y_1, \ldots, y_d) \in \mathbb{R}^d$ and $\varepsilon_\alpha y_\alpha \geqslant 0$ for any $1 \leqslant \alpha \leqslant d\} - \{x\}$. We show that if d is sufficiently large and $n \geqslant 1.76^d$ then there are $x \in P$ and $\varepsilon \in \{-1, 1\}^d$ such that the orthant $L_d(x, \varepsilon)$ contains at least two points of P.

1. Introduction

When A is a set, let $|A|$ or $\# A$ denote the cardinality of A. We call A n-set while $|A| = n$. Denote the αth-coordinate of x by x_α for $x \in \mathbb{R}^d$, $1 \leqslant \alpha \leqslant d$, i.e. $x = (x_1, x_2, \ldots, x_d)$. Let $l(d)$ be the largest number such that some $l(d)$-set $P \subset \mathbb{R}^d$ satisfies

$$|L_d(x, \varepsilon) \cap P| \leqslant 1 \text{ for every point } x \in P \text{ and every vector } \varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \in \{-1, 1\}^d, \tag{1}$$

where

$$L_d(x, \varepsilon) := \{x + y \mid y \in \mathbb{R}^d \text{ and } \varepsilon_\alpha y_\alpha \geqslant 0 \text{ for any } 1 \leqslant \alpha \leqslant d\} - \{x\}.$$

In other words, any $l(d) + 1$-pointset P in d-dimensional coordinate space has an orthant (whose origin is a point of P and whose faces are perpendicular or parallel to any axis) containing at least two points of P besides the origin. Let us take an alternate view of $l(d)$. Without using the famous result of Erdős and Szekeres [3], we can easily

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[1] Current address: Department of Computer Science and Information Mathematics, University of Electro-Communications, 1-5-1 Choufugaoka, Choufu City, Tokyo, 182 Japan.

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show that any sequence of five real numbers has a subsequence of length three, x, y, z, satisfying $(x-y)(y-z)\geqslant 0$. We see that any sequence of $(d-1)$-dimensional vectors of length $2^{2^{d-1}}+1$ has a subsequence x, y, z satisfying $(x_\alpha-y_\alpha)(y_\alpha-z_\alpha)\geqslant 0$ for any $1\leqslant\alpha<d$ from de Bruijn's theorem; see [1] or [5]. Note that the number $l(d)$ is the minimum number such that any sequence of $(d-1)$-dimensional vectors of length $l(d)+1$ has a subsequence x, y, z satisfying $(x_\alpha-y_\alpha)(y_\alpha-z_\alpha)\leqslant 0$ for any $1\leqslant\alpha<d$.

What is the number $l(d)$? This function was introduced in [4] for a containment problem with respect to a d-dimensional cuboid. By the pigeonhole principle, we can see that 2^d+1 is a trivial upper bound of $l(d)$. Clearly $l(2)=2$ and $l(3)=3$. Some computations yield $l(4)=4$. It was shown $7\leqslant l(5)$ and $0.21*1.47^d\leqslant l(d)\leqslant 2^{d-2}+1$ (for any $d\geqslant 2$) with a way to construct a set of at least $0.21*1.47^d$ points satisfying (1) in [4]. Enomoto proved $l(d)=o(2^d)$ in [2]. In this paper, we improve the upper bound with a different idea.

Theorem 1.1. *Some constant c satisfies*

$$l(d)<c\binom{d}{\lfloor d/4\rfloor}$$

for any $d\geqslant 1$.

We can show for some constant c' and sufficiently large d

$$\binom{d}{\lfloor d/4\rfloor}\leqslant c'\frac{1}{\sqrt{d}}\left(\frac{4}{3^{3/4}}\right)^d\leqslant 1.76^d$$

by using Stirling's formula $(\lim_{n\to\infty}(n!/\sqrt{2\pi}n^{n+1/2}e^{-n})=1)$.

2. Proof of Theorem 1.1

Let P be a set of n points in the d-dimensional Euclidean space satisfying (1). We show that n is small.

If the theorem holds when $n\equiv 2$, $d\equiv 0 \pmod 4$, then

$$n\leqslant c\binom{4(d_0+1)}{d_0+1}+n_e\leqslant c'\binom{4d_0+d_e}{d_0},$$

when $n=4n_0+2+n_e$, $d=4d_0+d_e$ (n_e, $d_e=0, 1, 2, 3$), where c and c' are constants. Thus it is sufficient to prove the theorem in the case of $n=4n_0+2$, $d=4d_0$.

Without loss of generality, we can suppose that any distinct $x, y\in P$ satisfy $x_\alpha\neq y_\alpha$ for any $1\leqslant\alpha\leqslant d$. Let

$$[x]_\alpha^+ := \{y\in P \mid x_\alpha<y_\alpha\},$$

$$[x]_\alpha^- := \{y\in P \mid x_\alpha>y_\alpha\}$$

for $1 \leqslant \alpha \leqslant d$, $x \in P$. Clearly,

$$P = [x]_\alpha^+ \cup [x]_\alpha^- \cup \{x\} \quad \text{(disjoint union)}. \tag{2}$$

Define a function $f: P \to \{0, 1, 2, \dots\}$ by $f(x) = \sum_{1 \leqslant \alpha \leqslant d} \min\{\#[x]_\alpha^+, \#[x]_\alpha^-\}$. We compute the average of $f(x)$:

$$\begin{aligned}\frac{1}{n}\sum_{x\in P} f(x) &= \frac{1}{n}\sum_{1\leqslant\alpha\leqslant d}\sum_{x\in P}\min\{\#[x]_\alpha^+, \#[x]_\alpha^-\}\\ &= \frac{1}{n}\sum_{1\leqslant\alpha\leqslant d}(0+1+2+\cdots+2n_0+2n_0+\cdots+2+1+0)\\ &= \frac{1}{n}2d\frac{2n_0(2n_0+1)}{2}\\ &= dn_0.\end{aligned}$$

Therefore, there exists $w \in P$ such that $f(w) \leqslant dn_0$. By symmetry we can suppose $\#[w]_\alpha^- \leqslant \#[w]_\alpha^+$ for any $1 \leqslant \alpha \leqslant d$ and

$$f(w) = \sum_{1\leqslant\alpha\leqslant d} \#[w]_\alpha^- \leqslant dn_0. \tag{3}$$

Define a function $\phi: P - \{w\} \to \{-3, 1\}^d$ by $\phi(x) := (\phi_1(x), \phi_2(x), \dots, \phi_d(x))$, where

$$\phi_\alpha(x) := \begin{cases} 1 & \text{if } x \in [w]_\alpha^+, \\ -3 & \text{if } x \in [w]_\alpha^-. \end{cases}$$

For distinct $x, y \in P$, if $\phi(x) = \phi(y)$, then $\{x, y\} \subset L_d(w, \varepsilon) \cap P$. Because P satisfies (1), the following claim holds.

Claim. *The function ϕ is injective.*

Define a function $\sigma: \{-3, 1\}^d \to \{0, \pm 1, \pm 2, \dots\}$ by $\sigma(\xi) := \sum_{1\leqslant\alpha\leqslant d} \xi_\alpha$ where $\xi = (\xi_1, \xi_2, \dots, \xi_d)$. When $\xi \in \{-3, 1\}^d$, $\sigma(\xi) = d - 4\#\{\alpha \mid \xi_\alpha = -3\}$. Let Ξ^+, Ξ^- denote the sets such that

$$\Xi^+ := \{\xi \in \{-3, 1\}^d \mid \sigma(\xi) \geqslant 0\} = \{\xi \in \{-3, 1\}^d \mid \#\{\alpha \mid \xi_\alpha = -3\} \leqslant d_0\},$$

$$\Xi^- := \{\xi \in \{-3, 1\}^d \mid \sigma(\xi) < 0\} = \{\xi \in \{-3, 1\}^d \mid \#\{\alpha \mid \xi_\alpha = -3\} > d_0\}.$$

Let us find an upper bound and a lower bound of $\sum_{x\in P-\{w\}} \sigma\phi(x)$. We have

$$\begin{aligned}\sum_{x\in P-\{w\}}\sigma\phi(x) &= \sum_{1\leqslant\alpha\leqslant d}\sum_{x\in P-\{w\}}\phi_\alpha(x)\\ &= \sum_{1\leqslant\alpha\leqslant d}\sum_{x\in[w]_\alpha^+}\phi_\alpha(x) + \sum_{1\leqslant\alpha\leqslant d}\sum_{x\in[w]_\alpha^-}\phi_\alpha(x)\\ &= \sum_{1\leqslant\alpha\leqslant d}(n-1-\#[w]_\alpha^-) - 3\sum_{1\leqslant\alpha\leqslant d}\#[w]_\alpha^-\\ &= (n-1)d - 4f(w)\\ &\geqslant d.\end{aligned} \tag{4}$$

Define $\Xi := \phi(P-\{w\}) \subset \{-3, 1\}^d$. Clearly, $|\Xi| = n-1$ by the above claim. So,

$$\begin{aligned}\sum_{x\in P-\{w\}} \sigma\phi(x) &= \sum_{\xi\in\Xi} \sigma(\xi)\\ &\leqslant \sum_{\xi\in\Xi^+} \sigma(\xi) + \sum_{\xi\in\Xi\cap\Xi^-} \sigma(\xi)\\ &\leqslant \sum_{0\leqslant i\leqslant d_0} \binom{d}{i}(d-4i) + (-4)|\Xi\cap\Xi^-|\\ &\leqslant 4\sum_{0\leqslant i\leqslant d_0} \binom{d}{i}(d_0-i) - 4(n-1-|\Xi^+|)\\ &\leqslant 4\left(\sum_{0\leqslant i\leqslant d_0} \binom{d}{i}(d_0-i+1)+1-n\right).\end{aligned} \tag{5}$$

By (4) and (5),

$$\begin{aligned} n &\leqslant \sum_{0\leqslant i\leqslant d_0} \binom{d}{i}(d_0-i+1)+1-d_0\\ &= \binom{d}{d_0}\left(\sum_{0\leqslant i\leqslant d_0} \frac{d_0!(d-d_0)!}{i!(d-i)!}(d_0-i+1)\right)+1-d_0\\ &\leqslant \binom{d}{d_0}\left(\sum_{0\leqslant j\leqslant d_0} \frac{j+1}{3^j}\right)+1-d_0\\ &< \frac{9}{4}\binom{d}{d_0}+1-d_0,\end{aligned}$$

for any $d = 4d_0 = 4, 8, 12, 16, \ldots$

Therefore the theorem holds.

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